Calculation of the steady flow through a curved tube using a new finite-difference method

By S. C. R. DENNIS

Department of Applied Mathematics, University of
Western Ontario, London, Canada

A numerical method is described which is suitable for solving the equations governing the steady motion of a viscous fluid through a slightly curved tube of circular cross-section but which is also applicable to the solution of any problem governed by the steady two-dimensional Navier–Stokes equations in the plane polar co-ordinate system. The governing equations are approximated by a scheme which yields finite-difference equations which are of second-order accuracy with respect to the grid sizes but which have associated matrices which are diagonally dominant. This makes them generally more amenable to solution by iterative techniques than the approximations obtained using standard central differences, while preserving the same order of accuracy.

The main object of the investigation is to obtain numerical results for the problem of steady flow through a curved tube which corroborate previous numerical work on this problem in view of a recent paper (Van Dyke 1978) which tends to cast doubt on the accuracy of previous calculations at moderately high values of the Dean number; this is the appropriate Reynolds-number parameter in this problem. The present calculations tend to verify the accuracy of previous results for Dean numbers up to 5000, beyond which it is difficult to obtain accurate results. Calculated properties of the flow are compared with those obtained in previous numerical work, with the predictions of boundary-layer theory for large Dean numbers and with the predictions of Van Dyke (1978).

1. Introduction

In a recent paper Dennis & Hudson (1978) described a method of solving the two-dimensional Navier–Stokes equations governing the motion of a viscous incompressible fluid in which a new type of approximation to the equation governing the transport of vorticity is made. In effect the basis of the method is not new since it is derived by an expansion of a method first introduced by Dennis (1960), but the essentially new point considered by Dennis & Hudson was the formulation of an approximation to the vorticity transport equation in which the matrix associated with the finite-difference equations is diagonally dominant in the sense defined by Varga (1962, p. 23). The method is of second-order accuracy with respect to the grid sizes, i.e. of the same order of accuracy as the method of approximation by central differences but, as is well known, central-difference approximations to second-order differential equations do not necessarily have associated matrices which are diagonally dominant.

Dennis & Hudson gave some illustrations of the convergence properties of iterative

methods of solution of the finite-difference equations in this particular formulation, but no comprehensive body of results was given in the study of a problem in which, as in the case to be considered here, there remain unresolved questions regarding the physical nature of the solution. In the present paper the results of some elaborate calculations obtained by the method are presented in the case of steady flow through a slightly curved tube of circular cross-section. At the same time it will be shown that the basic method of approximation considered by Dennis & Hudson in a rectangular co-ordinate system can be extended to problems in at least one other co-ordinate system, that is cylindrical co-ordinates. It is not completely obvious that the results given by Dennis & Hudson can be extended to other geometries or to other forms of the transport equations.

The steady flow of a viscous fluid through a curved tube of circular cross-section has received considerable attention from the point of view of theoretical, numerical and experimental treatments. The motion is assumed to be maintained by a constant pressure gradient G and to be fully developed so that the velocity components depend only on the space co-ordinates of a typical cross-section of the tube. The relevant dynamical similarity parameter for the motion is the Dean number $D = Ga^3(2a/L)^{\frac{1}{2}}/\mu\nu$, where a is the radius of the cross-section, L is the radius of the circle formed by the curved axis of the tube, μ is the viscosity of the fluid and ν the coefficient of kinematic viscosity. The earliest theoretical investigation of the problem was by Dean (1927, 1928) who assumed that the coiling ratio a/L was small and developed a perturbation solution of the equations of motion in powers of D giving the earlier terms, valid for small D, by exact analysis. Topakoglu (1967) retained higher-order terms in the ratio a/L giving a double series in powers of a/L and D. This series was extended to many more terms by Larrain & Bonilla (1970) by computer methods and the most recent work of this nature is the extension by Van Dyke (1978) of Dean's original series for the case $a/L = 0$ to 24 terms using computerized analysis. Van Dyke gives results for the friction ratio γ_c/γ_s, the ratio of the resistance coefficients in a curved and straight tube under the same pressure gradient, as a power series in D^4. By manipulation and re-structuring of this series in terms of a new variable

$$\epsilon = (K/576)^2/\{(0\cdot966\,858\,4)^{-1} + (K/576)^2\},$$

where $K = D^2/16$, a series for γ_c/γ_s is given which is claimed to be valid for all values of D.

This result of Van Dyke (1978) is in conflict with all previous numerical solutions of the Navier–Stokes equations in the case $a/L = 0$ for values of D greater than about 1000, although it agrees well enough with reliable solutions for smaller values of D. Numerical solutions for $a/L = 0$ have been given by McConalogue & Srivastava (1968), Truesdell & Adler (1970), Akiyama & Cheng (1971), Greenspan (1973), Austin & Seader (1973), Patankar, Pratap & Spalding (1974) and Collins & Dennis (1975). The papers of Truesdell & Adler and Austin & Seader also include the effect of finite a/L. Van Dyke's work is also in conflict with the predictions of boundary-layer theory as $D \to \infty$. Various investigations based on this theory have been given by Adler (1934), Barua (1963), Mori & Nakayama (1965) and Ito (1969). These theories do differ somewhat in detail but there is a remarkable consistency in the prediction of the tendency for γ_c/γ_s as $D \to \infty$. Further general studies have been made by Smith (1975, 1976). The major experimental investigations of the problem have been per-

pressure p and the velocity components are (U, V, W) in the directions of increase of (r', α, θ). A stream function $\psi(r', \alpha)$ exists for the motion and if new variables are introduced by the equations

$$r' = ar, \quad U = vu/a, \quad V = vv/a, \quad W = v(L/2a^3)^{\frac{1}{2}}w, \quad \psi = v\phi, \tag{1}$$

where v is the coefficient of kinematic viscosity then

$$u = r^{-1}\partial\phi/\partial\alpha, \quad v = -\partial\phi/\partial r. \tag{2}$$

On the assumption that a/L is small the Navier–Stokes equations of motion can be written as the three simultaneous equations

$$\nabla^2\phi = -\Omega, \tag{3}$$

$$\nabla^2 w + \frac{1}{r}\left(\frac{\partial\phi}{\partial r}\frac{\partial w}{\partial\alpha} - \frac{\partial\phi}{\partial\alpha}\frac{\partial w}{\partial r}\right) = -D, \tag{4}$$

$$\nabla^2\Omega + \frac{1}{r}\left(\frac{\partial\phi}{\partial r}\frac{\partial\Omega}{\partial\alpha} - \frac{\partial\phi}{\partial\alpha}\frac{\partial\Omega}{\partial r}\right) = w\left(\sin\alpha\frac{\partial w}{\partial r} + \frac{\cos\alpha}{r}\frac{\partial w}{\partial\alpha}\right), \tag{5}$$

where

$$\nabla^2 = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + r^{-2}\partial^2/\partial\alpha^2$$

and $D = Ga^3(2a/L)^{\frac{1}{2}}/\mu v$.

The velocity components must vanish at the surface of the tube $r = 1$ and hence

$$w = \phi = \partial\phi/\partial r = 0 \quad \text{when} \quad r = 1. \tag{6}$$

The flow is symmetrical about $\alpha = 0$ and $\alpha = \pi$ from which it follows that, for $0 \leqslant \alpha \leqslant \pi$,

$$\phi(r, -\alpha) = -\phi(r, \alpha), \quad w(r, -\alpha) = w(r, \alpha), \quad \Omega(r, -\alpha) = -\Omega(r, \alpha) \tag{7}$$

and in particular that

$$\phi = \Omega = 0, \quad \partial w/\partial\alpha = 0 \quad \text{when} \quad \alpha = 0, \pi. \tag{8}$$

In the numerical approach the equations (3)–(5) are solved subject to the boundary conditions by dividing the upper semi-circular region $r \leqslant 1$, $0 \leqslant \alpha \leqslant \pi$ into a grid formed by lines of constant r and constant α. The equations are then approximated in terms of finite differences at each grid point. The approximation to (3) is the customary central-difference approximation but to (4) and (5) a type of approximation which is in principle new is applied. It is based on a method given by Dennis & Hudson (1978) for the two-dimensional Navier–Stokes equations in rectangular co-ordinates. The method is such that individual forms of the governing equations in different co-ordinate systems must be considered separately.

3. Finite-difference approximations

We shall denote a typical grid point in the (r, α) grid system by (r_0, α_0) and number quantities at this point with subscript 0. Likewise quantities at the grid points $(r_0 + h, \alpha_0)$, $(r_0, \alpha_0 + k)$, $(r_0 - h, \alpha_0)$ and $(r_0, \alpha_0 - k)$ will be denoted by subscripts 1, 2, 3, 4 according to the notation of Southwell. Here h is the grid size in the radial direction and k that in the α direction. The finite-difference method follows a derivation of

Dennis (1960). In the case of equation (4) the method first proceeds by separating it into two equations which, after making use of (2), can be written

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - u\frac{\partial w}{\partial r} = A(r,\alpha) - D, \tag{9}$$

$$\frac{\partial^2 w}{\partial \alpha^2} - rv\frac{\partial w}{\partial \alpha} = -r^2 A(r,\alpha), \tag{10}$$

where $A(r,\alpha)$ is an unknown function. Along the line of constant α given by $\alpha = \alpha_0$ we put

$$w = F(r,\alpha_0)\exp\{-f(r,\alpha_0)\} \tag{11}$$

in (9), where

$$f(r,\alpha_0) = -\frac{1}{2}\int_{r_0}^{r} u(z,\alpha_0)\,dz. \tag{12}$$

The equation for F is, after simplification using (12),

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{2}\left(\frac{\partial u}{\partial r} + \frac{u}{r} - \frac{1}{2}u^2\right)F = \{A(r,\alpha_0) - D\}\exp\{f(r,\alpha_0)\}, \tag{13}$$

where all partial derivatives are to be evaluated at $\alpha = \alpha_0$. Similarly, along the line of constant r given by $r = r_0$ we put

$$w = G(r_0,\alpha)\exp\{-g(r_0,\alpha)\} \tag{14}$$

in (10), where

$$g(r_0,\alpha) = -\frac{1}{2}r_0\int_{\alpha_0}^{\alpha} v(r_0,z)\,dz. \tag{15}$$

The equation for G is found to be

$$\frac{\partial^2 G}{\partial \alpha^2} + \frac{1}{2}r_0\left(\frac{\partial v}{\partial \alpha} - \frac{1}{2}r_0 v^2\right)G = -r_0^2 A(r_0,\alpha)\exp\{g(r_0,\alpha)\}, \tag{16}$$

where all partial derivatives are to be evaluated at $r = r_0$.

We now approximate each of the equations (13) and (16) in terms of central differences at the point (r_0,α_0), where both equations must be applicable. The value $A(r_0,\alpha_0)$ is common to both and may therefore be eliminated. After replacing the functions F and G in terms of w using (11) and (14) respectively, we obtain the finite-difference approximation

$$\left(1 + \frac{h}{2r_0}\right)w_1 e^{f_1} + \left(1 - \frac{h}{2r_0}\right)w_3 e^{f_3} + \frac{\lambda^2}{r_0^2}(w_2 e^{g_2} + w_4 e^{g_4})$$
$$- \left[2 + \frac{2\lambda^2}{r_0^2} + \frac{1}{4}h^2(u_0^2 + v_0^2)\right]w_0 + h^2 D = 0, \tag{17}$$

where $\lambda = h/k$ and some simplification has occurred owing to the definitions of the velocity components in terms of ϕ. This approximation is second-order accurate in the sense that the truncation error on the right-hand side of (17) is of the same order as that which appears in the standard method of approximation of (4) by central differences, viz. the truncation error is $O(h^4) + O(h^2k^2)$. The approximation (17) holds at every grid point and thus defines a set of algebraic equations from which an approximation to w can be obtained throughout the computational domain, assuming that approximations to the other functions which appear in (17) can be determined from

(3) and (5). The matrix associated with (17) is not necessarily diagonally dominant as it stands. Diagonal dominance is a desirable property and we can, following the method of Dennis & Hudson (1978), obtain a form of (17) which is diagonally dominant and in which there is no further loss of order of accuracy in the truncation error beyond that which is already present.

If $u(r, \alpha)$ is expanded as a Taylor series in the direction of r at the point (r_0, α_0) and the result utilized in (12) we obtain

$$f(r, \alpha_0) = -\tfrac{1}{2} u_0 (r - r_0) - \frac{1}{4} \left(\frac{\partial u}{\partial r}\right)_0 (r - r_0)^2 - \frac{1}{12} \left(\frac{\partial^2 u}{\partial r^2}\right)_0 (r - r_0)^3 - \ldots, \tag{18}$$

from which f_1 and f_3 can be obtained in powers of h by putting $r = r_0 + h$ and $r = r_0 - h$ respectively. If we now expand the exponentials in the first two terms on the left-hand side of (17) in powers of their arguments and then in powers of h using the results for f_1 and f_3 derived from (18) it is found that

$$\left(1 + \frac{h}{2r_0}\right) w_1 e^{f_1} + \left(1 - \frac{h}{2r_0}\right) w_3 e^{f_3} = \left[1 + \frac{1}{8}\left\{\left(u_0 - \frac{2}{r_0}\right) u_0 - 2\left(\frac{\partial u}{\partial r}\right)_0\right\} h^2 + O(h^4)\right](w_1 + w_3)$$

$$+ \left[\frac{1}{2}\left(\frac{1}{r_0} - u_0\right) h + O(h^3)\right](w_1 - w_3). \tag{19}$$

It can similarly be shown by determining g_2 and g_4 as power series in k from (15) that

$$w_2 e^{g_2} + w_4 e^{g_4} = [1 + \tfrac{1}{8} r_0 \{r_0 v_0^2 - 2(\partial v/\partial \alpha)_0\} k^2 + O(k^4)](w_2 + w_4)$$

$$- [\tfrac{1}{2} r_0 v_0 k + O(k^3)](w_2 - w_4). \tag{20}$$

The truncation error terms already present in (17) are $O(h^4) + O(h^2 k^2)$ and thus there is no objection to omitting $O(h^4)$ terms from (19) and $O(k^4)$ terms from (20). In making a further reduction it may be noted that $w_1 - w_3 = O(h)$, $w_2 - w_4 = O(k)$ and further that

$$w_1 + w_3 = 2w_0 + O(h^2), \quad w_2 + w_4 = 2w_0 + O(k^2). \tag{21}$$

With the aid of these relationships in (19) and (20) and the omission of error terms whose order is at least equal to the order of the error terms already inherent in equation (17) it is found, after some simplification arising from the use of (2), that (17) can finally be expressed as the approximation

$$c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_4 - c_0 w_0 + h^2 D = 0, \tag{22}$$

where

$$\left.\begin{aligned}
&c_1 = 1 + h/2r_0 - \tfrac{1}{2} u_0 h + u_0^2 h^2/8, \quad c_2 = \lambda^2 (1/r_0^2 - \tfrac{1}{2} v_0 k/r_0 + v_0^2 k^2/8), \\
&c_3 = 1 - h/2r_0 + \tfrac{1}{2} u_0 h + u_0^2 h^2/8, \quad c_4 = \lambda^2 (1/r_0^2 + \tfrac{1}{2} v_0 k/r_0 + v_0^2 k^2/8), \\
&c_0 = 2 + 2\lambda^2/r_0^2 + \tfrac{1}{4} h^2 (u_0^2 + v_0^2).
\end{aligned}\right\} \tag{23}$$

The approximation (22) is applicable at all grid points for which $r_0 \neq 0$. The smallest value of r_0 for which it is applied is $r_0 = h$ and the smallest value of $1 - h/2r_0$ is thus $\tfrac{1}{2}$. For $r_0 \geq h$ it may then be demonstrated that $c_1 \geq 0$, $c_3 \geq 0$ for all values of $u_0 h$ and also that $c_2 > 0$, $c_4 > 0$ for all values of $v_0 k$. Since

$$c_1 + c_2 + c_3 + c_4 = c_0 \tag{24}$$

also, it follows that the matrix associated with the set of equations (22) is diagonally dominant in the sense defined by Varga (1962, p. 23). Moreover, if we collect together

all the terms involving h^2 in the first four terms of (22) and apply (21) it is found that with an error $O(h^4) + O(h^2 k^2)$ they cancel the corresponding terms in h^2 which appear in $c_0 w_0$ in (22) and we are left with the standard central-difference approximation to (4). Thus (22) is of the same order of accuracy with respect to the grid sizes as the approximation to (4) by central differences but the actual error terms are obviously quite different.

When $r = 0$ there is a set of grid points as α varies from $\alpha = 0$ to $\alpha = \pi$ at which w has the same constant value which must be determined as part of the solution. The finite-difference equations (22) are not directly applicable here but we can follow the procedure adopted by Collins & Dennis (1975) and take rectangular co-ordinates (x, y) with origin at $r = 0$ and the axis of x along some particular line $\alpha = \alpha_0$. Equation (4) then becomes

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial \phi}{\partial y} \frac{\partial w}{\partial x} + \frac{\partial \phi}{\partial x} \frac{\partial w}{\partial y} = -D. \tag{25}$$

This equation may then be approximated directly by the method of Dennis & Hudson (1978) to give the finite-difference equation

$$\begin{aligned}
(1 - \tfrac{1}{2} h u_0 + \tfrac{1}{8} h^2 u_0^2)\, w(h, \alpha_0) &+ (1 - \tfrac{1}{2} h v_0 + \tfrac{1}{8} h^2 v_0^2)\, w(h, \tfrac{1}{2}\pi + \alpha_0) \\
+ (1 + \tfrac{1}{2} h u_0 + \tfrac{1}{8} h^2 u_0^2)\, w(h, \pi - \alpha_0) &+ (1 + \tfrac{1}{2} h v_0 + \tfrac{1}{8} h^2 v_0^2)\, w(h, \tfrac{1}{2}\pi - \alpha_0) \\
&- \{4 + \tfrac{1}{4} h^2 (u_0^2 + v_0^2)\}\, w(0, \alpha_0) + h^2 D = 0. \tag{26}
\end{aligned}$$

Here (u_0, v_0) denote the radial and transverse components of velocity at the point $(0, \alpha_0)$ and the grid values of w in (26) refer to the polar co-ordinate system centred at $r = 0$. Equation (26) applies without modification if $0 \leqslant \alpha_0 \leqslant \tfrac{1}{2}\pi$ since all points referenced lie in the computational domain, but if $\tfrac{1}{2}\pi < \alpha_0 \leqslant \pi$ then values of w at grid points outside the computational field must be replaced using $w(h, -\alpha) = w(h, \alpha)$.

The two sets of equations (22) and (26) may together be used to obtain approximations to w at all grid points in the domain $0 \leqslant r < 1$, $0 \leqslant \alpha \leqslant \pi$. At grid points on either $\alpha = 0$ or $\alpha = \pi$ one of the grid values of w in (22) lies outside the domain and thus is dealt with by using the relation $w_2 = w_4$ which corresponds to the second condition of (7). The necessary values of (u_0, v_0) in (23) and (26) are calculated from (2) using the central-difference approximations

$$u_0 = (\phi_2 - \phi_4)/2kr_0, \quad v_0 = -(\phi_1 - \phi_3)/2h. \tag{27}$$

Since $r_0 = 0$ in the case of (26) we then replace kr_0 by h in (27). At grid points on $\alpha = 0$ and $\alpha = \pi$ use is made of the relation $\phi_2 = -\phi_4$ which follows from the first condition of (7). This condition along with the condition $\phi_3 = -\phi_1$ is also used in calculating the values of (u_0, v_0) required in the set of equations (26) which hold on $r = 0$. This set holds at all grid points on $r = 0$ and must yield a set of values of $w(0, \alpha)$ which are equal since there is a unique value of w at $r = 0$. This was confirmed to a high degree of accuracy in all the computations.

Approximations to the equations (3) and (5) are needed at all grid points internal to the domain $0 \leqslant r \leqslant 1$, $0 \leqslant \alpha \leqslant \pi$. For the equation (3) the standard central-difference approximation

$$\left(1 + \frac{h}{2r_0}\right) \phi_1 + \left(1 - \frac{h}{2r_0}\right) \phi_3 + \frac{\lambda^2}{r_0^2} (\phi_2 + \phi_4) - 2 \left(1 + \frac{\lambda^2}{r_0^2}\right) \phi_0 + h^2 \Omega_0 = 0 \tag{28}$$

is used. For (5) a similar method to that described for (4) is used, with the right-hand side approximated by standard central-difference formulae. This gives the set of equations

$$c_1 \Omega_1 + c_2 \Omega_2 + c_3 \Omega_3 + c_4 \Omega_4 - c_0 \Omega_0 - \tfrac{1}{2} h w_0 \left\{ (w_1 - w_3) \sin \alpha_0 + \frac{\lambda}{r_0} (w_2 - w_4) \cos \alpha_0 \right\} = 0,$$

(29)

where the coefficients c_n are those defined by (23). The boundary conditions for (28) are that ϕ must vanish on all boundaries of the semi-circular domain while for (29) Ω must vanish on $r = 0$, $\alpha = 0$ and $\alpha = \pi$, but must be calculated at grid points on $r = 1$ by making use of the conditions $\phi = \partial\phi/\partial r = 0$ at $r = 1$ given in (6) together with the finite-difference equations (28).

There are several methods of carrying out this calculation. Collins & Dennis (1975) used the customary central-difference approximation to the condition $\partial\phi/\partial r = 0$ at $r = 1$ which, in conjunction with (28) and the condition $\phi = 0$ at $r = 1$ gives the approximation

$$\Omega(1, \alpha) = -2\phi(1 - h, \alpha)/h^2.$$

(30)

The main objection to this formula, which is quite commonly used, is that the truncation error on the right-hand side of (30) is $O(h)$ whereas the truncation error in the remainder of the finite-difference equations used is of the second order. This loss of accuracy has implications when an attempt is made to extrapolate the properties of the computed solutions to zero grid size. It is therefore desirable to improve the accuracy of the calculation of $\Omega(1, \alpha)$. This may be done by using a method which in principle is due to Woods (1954). The application of this method to problems similar to the present one was considered by Collins & Dennis (1976a, b). The only difference in the present case is that the Laplacian operator in (3) is the operator in the polar co-ordinate system. A simple extension of the results given by Collins & Dennis (1976a) gives the result

$$(2 + h)\,\Omega(1, \alpha) = -6\phi(1 - h, \alpha)/h^2 - \Omega(1 - h, \alpha)$$

(31)

applicable to the present case. The error term on the right-hand side of (31) is now $O(h^2)$ which brings the calculation of the boundary condition for $\Omega(1, \alpha)$ up to the same general order of accuracy inherent in the remainder of the finite-difference equations.

4. Numerical procedures

The use of (31) gives an improvement in the present method over that used by Collins & Dennis (1975). It is also unnecessary to employ the elaborate procedure used by Collins & Dennis (1975) of first obtaining a numerical solution for a given D using finite-difference equations of only first-order accuracy to approximate (4) and (5) and subsequently improving the accuracy by an iterative correction procedure. In the present formulation of the problem the sets of finite-difference equations (22), (26), (28) and (29) can be solved subject to their boundary conditions by an iterative procedure which converges satisfactorily for all $D \leqslant 5000$. The satisfactory convergence is due to the fact that each individual set of difference equations for w, ϕ and Ω has,

when treated as a set of linear algebraic equations governing the appropriate function, an associated matrix which is diagonally dominant. Thus (Varga 1962) the successive over-relaxation procedure converges for each individual set for a well-defined range of the relaxation factor and this facilitates the convergence of the iterative procedure in which each set of equations is solved one after the other. In general, a solution of second-order accuracy can thus be obtained by the present method with comparable efficiency to that required to obtain a solution of first-order accuracy by means of upwind–downwind schemes such as that used by Collins & Dennis (1975) to obtain a first approximation.

The successive over-relaxation procedure was employed for solving all the finite-difference equations. For the set of equations (22) this may be written

$$w_0^{(m+1)} = (1-\omega)\,w_0^{(m)} + \omega\{(c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_4 + h^2 D)/c_0\}^{(m)}, \tag{32}$$

where the superscripts denote successive iterates and ω is the relaxation factor. All quantities in the last expression on the right-hand side of (32) are evaluated from the most recent data. Similar equations correspond to (26), (28) and (29) and the solution procedure was carried out as follows. From an initial approximation to w, ϕ and Ω at all grid points, including boundary points, a complete iteration over all grid points for which $h \leqslant r \leqslant 1-h$, $0 \leqslant \alpha \leqslant \pi$ was performed using (32) followed by an iteration over all points for which $r = 0$, $0 \leqslant \alpha \leqslant \pi$ using the equation corresponding to (26). The approximation to w thus obtained was substituted in (29) and one complete iteration corresponding to this set of equations carried out for the internal grid points $h \leqslant r \leqslant 1-h$, $k \leqslant \alpha \leqslant \pi-k$. The boundary condition for Ω at $r = 1$ is kept fixed at the most recent distribution calculated from (31) during this iteration. The approximation to Ω determined at internal grid points is then introduced into (28) and a complete iteration determines an approximation to ϕ. Then an application of (31) determines a new set of boundary values for Ω. This sequence of operations defines one major iteration. The major iterations are repeated until convergence, defined by the test

$$\Sigma\left|1 - w^{(m+1)}(r,\alpha)/w^{(m)}(r,\alpha)\right| < 0.000\,05, \tag{33}$$

where the summation extends to all grid points such that $0 \leqslant r < 1$, $0 \leqslant \alpha \leqslant \pi$.

In practice two sums similar to the left-hand side of (33) were computed for Ω and ϕ except that here the sum is extended only to grid points where the appropriate function is given to be non-zero. All three sums were tabulated from time to time during the iterative procedure and their rate of decrease observed. Several representative properties of the solution were tabulated along with these sums and their convergence to limits observed. In this way it was ensured that all quantities had converged to limits to a very high degree of accuracy. Solutions were carried out for $D = 96$, 500, 1000, 3000, 4000 and 5000. For $D \leqslant 1000$ it was possible to accelerate convergence by using values of ω up to $\omega = 1.3$ in (32) and the similar equations corresponding to all the sets of difference equations. For $D \geqslant 3000$ it was not possible to take $\omega > 1$ for any of these sets, but $\omega = 1$ was always possible in solving the sets (28) and (29). For $D = 4000$ and 5000 it was found to be suitable to take $\omega < 1$ in (32) and in the similar procedure corresponding to (26). This decreases the forcing effect of the axial velocity component derived from (22) and (26) on the approximation to Ω obtained from the set of equations (29) and decreases the tendency of the whole sequence of iterative procedures to diverge. However, the convergence is extremely

slow for $D = 5000$ and it would be very difficult to proceed to $D > 5000$. In every case over the range of D considered it was possible to use (31) directly to calculate new boundary values for $\Omega(1, \alpha)$ without any under-relaxation. This was not possible in the corresponding procedure used by Collins & Dennis (1975).

The main extension of the present calculations beyond those of Collins & Dennis (1975) is that solutions could be obtained for three different grid structures for each value of D, namely

$$(a)\; h = 1/10, \quad k = \pi/10; \quad (b)\; h = 1/20, \quad k = \pi/20; \quad (c)\; h = 1/40, \quad k = \pi/40. \quad (34)$$

Collins & Dennis (1975) were able to employ only the one pair of grid sizes $h = 1/40$, $k = \pi/72$ for $D \geqslant 3500$ and although a smaller grid size in the α direction was used the evidence of the calculated results presented in the following section is that this was not necessary. Since the ratio h/k is kept constant in each of the three grid structures given by (34) it is possible to apply h^2 extrapolation to many of the physical properties computed from the solutions, particularly global properties such as the friction ratio γ_c/γ_s. This procedure gave extremely consistent results in the investigation of steady flow through a curved tube of triangular cross-section by Collins & Dennis (1976b). In the present case we can denote a given property, considered as a function of the grid structure only, by $P(h, k)$ and let P_1, P_2, P_3 be its values corresponding to the respective grids (a), (b), (c) in (34). Then the extrapolated value P_{12} between P_1 and P_2 is given by

$$P_{12} = \tfrac{1}{3}(4P_2 - P_1) \qquad (35)$$

with a similar formula for the extrapolated value P_{23} between P_2 and P_3. On the assumption that the error proceeds in even powers of h and k it is then possible to obtain a further extrapolated value P_{123} from P_{12} and P_{23} in the form

$$P_{123} = (16P_{23} - P_{12})/15. \qquad (36)$$

These results will be used during the analysis of the major properties of the flow which follows.

5. Calculated results

The predictions of Van Dyke (1978) suggest that all previous numerical solutions in the case of indefinitely small a/L, which is the assumption made in the present work, are seriously in error. This conclusion is drawn from a comparison of one property only presented in Van Dyke's results in the same case $a/L \to 0$, namely the friction ratio. This can easily be calculated in terms of a Reynolds number κ which it is customary to use in experimental measurements. This is defined by

$$\kappa = 2aW_m(a/L)^{\frac{1}{2}}/\nu, \qquad (37)$$

where W_m is the mean axial velocity of the flow through the tube. It is found to be given by (Collins & Dennis 1975)

$$\kappa = \frac{2\sqrt{2}}{\pi} \int_0^1 \int_0^\pi wr \, dr \, d\alpha \qquad (38)$$

and it may then be shown that

$$\gamma_s/\gamma_c = f(D) = 4\sqrt{2}\,\kappa D^{-1}. \qquad (39)$$

D	Grid	κ	γ_c/γ_s	ϕ_M	w_M	w_c
96	(a)	16·56	1·025	0·995	23·32	22·41
	(b)	16·58	1·024	0·990	23·35	22·44
	(c)	16·58	1·023	0·988	23·35	22·44
500	(a)	64·86	1·363	6·161	82·35	63·09
	(b)	65·86	1·342	6·123	83·39	63·63
	(c)	66·14	1·336	6·105	83·67	63·78
1000	(a)	109·8	1·609	9·483	135·7	96·76
	(b)	113·0	1·564	9·224	139·9	98·61
	(c)	114·0	1·551	9·185	141·1	99·16
3000	(a)	239·9	2·211	16·66	293·3	192·4
	(b)	251·6	2·108	15·79	310·5	200·3
	(c)	255·6	2·075	15·76	314·8	203·5
4000	(a)	291·1	2·430	19·03	352·0	230·5
	(b)	308·1	2·295	17·82	378·5	242·7
	(c)	314·0	2·252	17·82	384·4	246·8
5000	(a)	336·4	2·628	21·07	403·9	264·5
	(b)	360·0	2·455	19·48	440·1	282·0
	(c)	368·1	2·401	19·49	448·0	287·9

TABLE 1. Variation of flow properties with grid.

The left-hand member of (39) is the reciprocal of the quantity analysed in detail by Van Dyke (1978). Both κ and γ_c/γ_s depend only on D but, in the case of numerical solutions, estimates of them will vary with the grid used. In the first instance we can test the accuracy of the present computations by examining the variation of these and other properties with grid size.

In table 1 we show calculated values of κ, γ_c/γ_s and three other quantities for the three grids used for each value of D. The integral on the right of (38) was evaluated by the two-dimensional form of Simpson's rule. The quantity ϕ_M is the maximum value of ϕ over the cross-section whose position and magnitude was determined by the method described by Collins & Dennis (1976a). The quantity w_M is the maximum value of the dimensionless axial velocity component. This occurs on the axis of symmetry of the cross-section $\alpha = 0$ and its position and magnitude were determined by several accurate methods of estimating the position of the zero of $\partial w/\partial r$ on $\alpha = 0$, all of which agreed to good accuracy. Finally, the quantity w_c is the value of the dimensionless axial velocity component at the centre of the cross-section. All these quantities appear to be approaching definite limits as the size of the grid is reduced. It is not possible to make a direct comparison with the results of Collins & Dennis (1975) for equivalent grids since Collins & Dennis used the grids: $h = 1/10$, $k = \pi/18$; $h = 1/20$, $k = \pi/36$; $h = 1/40$, $k = \pi/72$. However, there is a very clear consistency in the trend of both sets of calculations where comparison is possible. In particular, the values of κ and γ_c/γ_s obtained using the grid $h = 1/40$, $k = \pi/40$ in the present work are in almost exact agreement with those obtained by Collins & Dennis (1975) using the grid $h = 1/40$, $k = \pi/72$. Collins & Dennis (1975) did not publish results for w_c

	Van Dyke (1978)		Collins & Dennis (1975)		Present	
D	κ	γ_c/γ_s	κ	γ_c/γ_s	κ	γ_c/γ_s
96	16·58	1·024	16·6	1·023	16·58	1·023
500	65·70	1·345	66·1	1·337	66·23	1·335
1000	114·4	1·545	114·0	1·550	114·3	1·546
2000	199·2	1·774	190·9	1·852	—	—
3000	—	—	—	—	257·1	2·063
3500	311·8	1·985	285·8	2·165	—	—
4000	—	—	—	—	316·2	2·237
5000	414·7	2·131	369·5	2·392	370·9	2·383

TABLE 2. Comparison of results for the Reynolds number and friction ratio.

but the values obtained by them using the grid $h = 1/40$, $k = \pi/72$ were found to be

D	96	500	1000	2000	3500	5000
w_c	22·44	63·70	99·00	154·7	224·7	287·2

giving excellent comparison with the results for $h = 1/40$, $k = \pi/40$ in table 1.

More accurate estimates of κ and γ_c/γ_s can be obtained by applying the h^2-extrapolation formulae (35) and (36) to the results given in table 1. The final estimates obtained from (36) are shown in table 2, where they are compared with the results of Van Dyke (1978, p. 140) and the calculations of Collins & Dennis (1975, p. 153) for the grid $h = 1/40$, $k = \pi/72$. These results of Collins & Dennis are probably more strictly comparable with the results in table 1 for the grid (c) than with the extrapolated results in table 2 but in any case there is little difference in all three sets of results and one must conclude that an error of the magnitude suggested by Van Dyke is out of the question. The method of h^2 extrapolation is not strictly applicable to the results of Collins & Dennis (1975) because of their use of (30) for calculating $\Omega(1, \alpha)$ which introduces an error of order h into this part of the calculation rather than the error of order h^2 introduced by use of (31). Notwithstanding this it is of interest to note that if we apply h^2 extrapolation to the results at $D \leqslant 1000$ for ϕ_M and w_M in table 1 and compare the results with a similar h^2 extrapolation of the results of Collins & Dennis (1975, p. 143) obtained using the somewhat different grid structures noted above, we obtain almost exact coincidence between the final extrapolated estimates derived from (36) for all $D \leqslant 1000$ in both cases. This gives yet another consistency check between two sets of results in which the truncation errors present in the majority of the approximating sets of finite-difference equations are different.

The nature of the discrepancy between the results for γ_c/γ_s given by Collins & Dennis (1975) and Van Dyke (1978) as D increases can in essence be expressed by the fact that Collins & Dennis found that their results were consistent with the asymptotic expression

$$\gamma_c/\gamma_s \sim 0\cdot1028\kappa^{\frac{1}{2}} + 0\cdot380 \quad \text{as} \quad D \to \infty \tag{40}$$

whereas Van Dyke asserts from his results that

$$\gamma_c/\gamma_s \sim 0\cdot47136\kappa^{\frac{1}{4}} \quad \text{as} \quad D \to \infty. \tag{41}$$

D	$D^{-\frac{2}{3}}\kappa$	$D^{-\frac{1}{3}}\gamma_c/\gamma_s$	$D^{-\frac{1}{3}}\phi_M$	$D^{-\frac{2}{3}}w_M$	$D^{-\frac{2}{3}}w_c$	$D^{\frac{1}{3}}(1-d)$
96	0·791	0·2235	0·216	1·114	1·071	3·65
500	1·051	0·1681	0·768	1·330	1·013	3·47
1000	1·143	0·1546	0·917	1·415	0·993	3·68
3000	1·236	0·1430	1·094	1·520	0·980	3·74
4000	1·255	0·1409	1·125	1·533	0·985	3·79
5000	1·269	0·1394	1·142	1·541	0·992	3·85

TABLE 3. Final properties of the flow obtained by h^2 extrapolation.

The leading term of the asymptotic formula (40) is consistent with that predicted by all of the existing boundary-layer theories. Despite variations in the individual versions of the boundary-layer approach to the problem, they all predict a variation of the form

$$\gamma_c/\gamma_s \sim A\kappa^{\frac{1}{2}} \quad \text{as} \quad D \to \infty \tag{42}$$

with somewhat different estimates of the constant A given as $A = 0\cdot1064$, $0\cdot09185$, $0\cdot1080$ and $0\cdot1033$ by Adler (1934), Barua (1963), Mori & Nakayama (1965) and Ito (1969) respectively. Collins & Dennis obtained the result (40) by making the assumption

$$D^{\frac{1}{3}}\gamma_s/\gamma_c \sim a + bD^{-\frac{1}{3}} \tag{43}$$

as $D \to \infty$ and estimating the constants a and b graphically from a plot of $D^{\frac{1}{3}}\gamma_s/\gamma_c$ against $D^{-\frac{1}{3}}$. This gave the values $a = 8\cdot12$, $b = -16\cdot7$ and then some manipulation of (43) yielded (40). In the case of the present results we have estimated a and b by fitting (43) to results obtained at successive pairs of values of D and observing the tendency of a and b as D increases. Calculated values of $\kappa D^{-\frac{2}{3}}$ based on the extrapolated results of table 2 are given in table 3. Apart from a factor, these give the left-hand side of (43). They appear to be approaching a limit as D increases. The fitted values of a and b are also found to approach limits quite smoothly and the expression

$$D^{\frac{1}{3}}\gamma_s/\gamma_c \sim 8\cdot19 - 17\cdot4D^{-\frac{1}{3}} \tag{44}$$

fits the results for $D \geqslant 1000$ very accurately. The suggested asymptotic relationship between D and κ for large D from the present results is

$$D^{\frac{1}{3}} \sim 0\cdot831\kappa^{\frac{1}{2}} + 1\cdot06 \tag{45}$$

and the corresponding result to (40) is

$$\gamma_c/\gamma_s \sim 0\cdot1015\kappa^{\frac{1}{2}} + 0\cdot388. \tag{46}$$

The slight differences between the results (44) and (46) and the corresponding predictions of Collins & Dennis (1975) are due to the use of the presumably more accurate data obtained by h^2 extrapolation in the present work.

On the basis of the prediction (41) Van Dyke (1978) has dismissed the structure of the solution as $D \to \infty$ given by the various boundary-layer analyses as incorrect, although no suggestion as to the correct structure is made. The conventional boundary-layer models suggest that the boundary-layer thickness near the tube wall is proportional to $D^{-\frac{1}{3}}$ with $\phi = O(D^{\frac{1}{3}})$, $w = O(D^{\frac{2}{3}})$ and $\Omega = O(D)$ in the boundary layer. In order to match the conditions in the boundary layer the axial velocity component in the inviscid core flow outside the boundary-layer region must also be $O(D^{\frac{2}{3}})$. Although
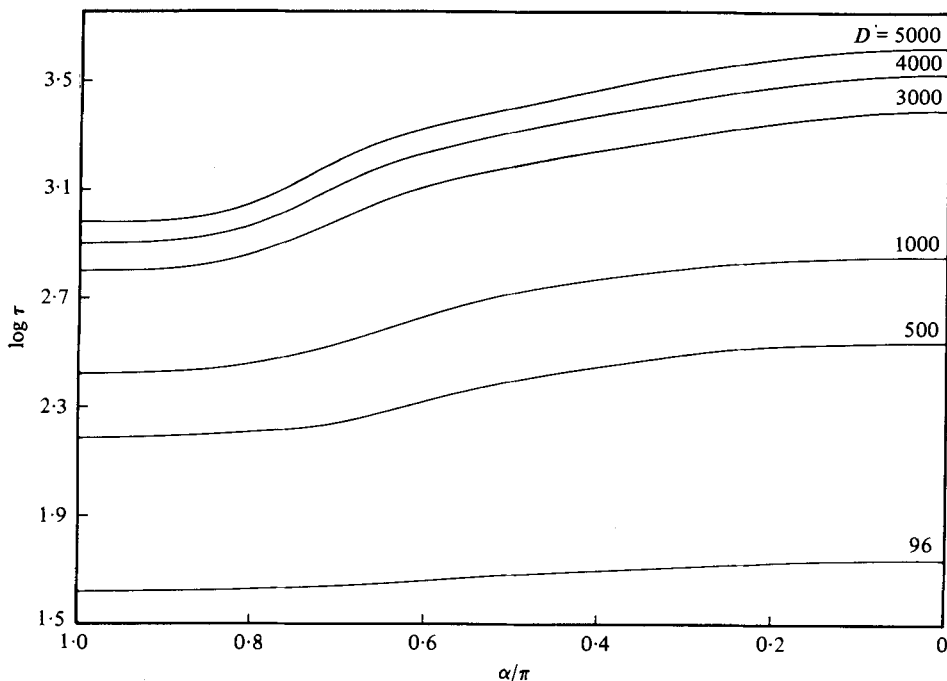
FIGURE 2. Distribution of the axial component of skin
friction over the wall of the tube.

any attempt to confirm this structure using the present numerical solutions must necessarily be viewed as tentative, there is certainly a suggestion of its validity. In table 3 we give some results for typical properties of the flow scaled according to the conventional boundary-layer theory. In every case the results represent the final value obtained using (36) by h^2 extrapolation from solutions obtained on the three grids given by (34). The quantities ϕ_M, w_M and w_c correspond to those given in table 1 and d is the dimensionless distance of the location of w_M from the centre of the tube. Thus the quantity $D^{\frac{1}{2}}(1-d)$ gives a measure of the boundary-layer thickness at the outer bend of the tube in terms of a co-ordinate normal to the wall scaled in accordance with the conventional theory.

It is evident that the results of table 3 are not inconsistent with the boundary-layer structure proposed by the theories. It is possible to attempt to estimate asymptotic formulae for these properties as $D \to \infty$ by representing each by an expression having the same form as the right-hand side of (43) and estimating the constants as the limits for increasing D of a process of fitting the expression to results obtained at successive pairs of values of D. In every case this procedure shows some semblance of an approach to a limiting behaviour as D increases but in most cases this limit, if it exists, has not been reached at $D = 5000$. In the case of the property $D^{-\frac{3}{4}}w_M$ we can quote the possible limiting formula

$$D^{-\frac{3}{4}}w_M \sim 1\cdot645 - 1\cdot78D^{-\frac{1}{4}} \tag{47}$$

as $D \to \infty$ which fits the results for $D \geqslant 3000$ almost exactly and is less than 4% in error at $D = 1000$. The most surprising behaviour in table 3 is that of $D^{-\frac{3}{4}}w_c$ which

D	$-D^{-1}\Omega_M$	$-D^{-1}\left(\dfrac{\partial\Omega}{\partial\alpha}\right)_{r=1,\alpha=0}$	$D^{-1}\left(\dfrac{\partial\Omega}{\partial\alpha}\right)_{r=1,\alpha=\pi}$	$D^{-1}\tau(0)$	$D^{-1}\tau(\pi)$
96	0·239	0·311	0·176	0·572	0·441
500	0·550	0·664	0·181	0·691	0·308
1000	0·619	0·775	0·158	0·722	0·267
3000	0·663	1·033	0·121	0·812	0·211
4000	0·682	1·036	0·114	0·817	0·199
5000	0·698	1·026	0·110	0·818	0·190

TABLE 4. Representative properties of the transverse and axial components
of skin friction at the wall of the tube.

first falls to a minimum with increasing D and then starts to rise. It is unlikely that this behaviour is due to numerical error. The results of table 1 indicate a quite consistent tendency towards a limit for all D as the grid sizes h and k are reduced; and the h^2-extrapolation process gives convincing results. We have already noted the excellent agreement for this property between the present results using the grid $h = 1/40$, $k = \pi/40$ and those of Collins & Dennis (1975) using the grid $h = 1/40$, $k = \pi/72$. For these respective grids the smallest observed value of the property in question in the present calculations is $D^{-\frac{3}{2}}w_c = 0\cdot976$ at $D = 3000$ rising to $D^{-\frac{3}{2}}w_c = 0\cdot985$ at $D = 5000$, whereas Collins & Dennis find $D^{-\frac{3}{2}}w_c = 0\cdot975$ for both $D = 2000$ and 3500 rising to $D^{-\frac{3}{2}}w_c = 0\cdot982$ at $D = 5000$. Further confirmation has been obtained in independent calculations by Dennis & Ng (1980) using a method of series truncation in which the co-ordinate α is eliminated. A similar minimum of $D^{-\frac{3}{2}}w_c$ is found of approximately the same magnitude but there is a marginally greater rise in $D^{-\frac{3}{2}}w_c$ as $D = 5000$ is approached.

The main properties of the flow given graphically by Collins & Dennis (1975) are confirmed to good accuracy in the present calculations and need not be described further. We shall, however, give some results for the distributions of the axial and transverse components of skin friction over the wall of the tube, which were not given in detail by Collins & Dennis. The dimensionless axial component of skin friction at the wall can be measured by the coefficient

$$\tau(\alpha) = -(\partial w/\partial r)_{r=1}. \tag{48}$$

In figure 2 we give results for $\log\tau$ as a function of α over the range of D of the calculations. The derivative on the right-hand side of (48) was evaluated using sufficient terms of a series of backward differences of w based on $r = 1$. The similarity of the curves for $D \geqslant 3000$ over the entire range of α is evident and the separation between the curves is quite consistent with the conventional boundary-layer notion that $D^{-1}\tau(\alpha)$ tends to become a function of α alone over the wall of the tube as D increases. Some precise numerical information for $D^{-1}\tau(0)$ and $D^{-1}\tau(\pi)$ is given in table 4. The dimensionless transverse component of skin friction at the wall is measured by $\Omega(1,\alpha)$. Distributions of $D^{-1}\Omega(1,\alpha)$ are shown in figure 3. The curve for $D = 4000$ has been omitted but it generally lies between those for $D = 3000$ and 5000 over the entire range of α.

Some representative values of the axial and transverse components of skin friction at the wall are given in table 4. The quantity Ω_M is the value of Ω with the greatest
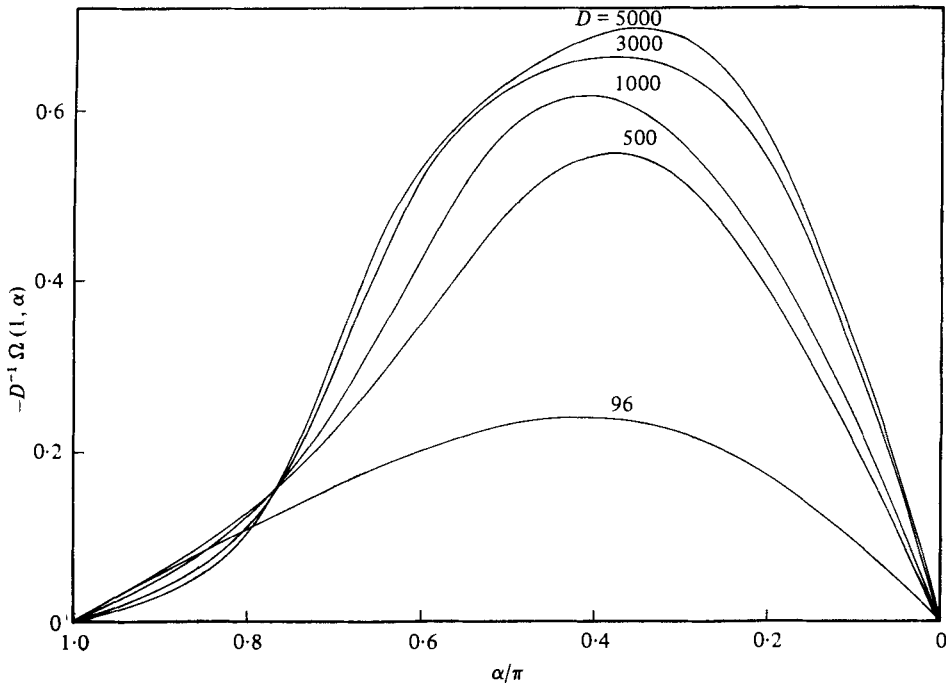
FIGURE 3. Distribution of the transverse component of
skin friction over the wall of the tube.

numerical magnitude throughout the computational domain; this occurs on the wall
of the tube. It was obtained from the solutions for each of the three grids (34) by
locating the zero of $(\partial\Omega/\partial\alpha)_{r=1}$ and then applying an interpolation procedure. The
results presented in table 4 for Ω_M are the final extrapolated values obtained from
the values on the three grids using (36). The values of $D^{-1}(\partial\Omega/\partial\alpha)_{r=1}$ at $\alpha = 0$ and
$\alpha = \pi$ in table 4, which measure the slopes of the curves in figure 3 at the corresponding
points, were also obtained by extrapolation from values on the three grids using (36).
All three of the properties of the transverse component of skin friction given in table 4
are not inconsistent with the tendency predicted by boundary-layer theory in that
they should tend to constants with increasing D. The slight fall in the magnitude of
$D^{-1}(\partial\Omega/\partial\alpha)_{r=1}$ at $\alpha = 0$ as D increases from 4000 to 5000 is due to inadequacy of the
extrapolation procedure for this particular quantity for large D. This fall does not
take place in the magnitudes of the corresponding quantities obtained from the
$h = 1/40$, $k = \pi/40$ grid but the extrapolation procedure is influenced by inaccurate
values of $(\partial\Omega/\partial\alpha)_{r=1}$ at $\alpha = 0$ on the coarsest grid when D is large.

Values of the axial component of skin friction at $\alpha = 0$ and $\alpha = \pi$ are given in table 4.
Some care was taken in their evaluation since they provide useful reference quantities
for comparison with other work on the problem. Three independent methods of
calculating the derivative in (48) were used. In the first, as mentioned previously,
a number of terms of a series of backward differences of w based on $r = 1$ was used.
In the second method the simplest central-difference approximation

$$(\partial w/\partial r)_{r=1} = \{w(1 + h, \alpha) - w(1 - h, \alpha)\}/2h$$

was used with the external value $w(1+h, \alpha)$ calculated from (22) with

$$r_0 = 1, \quad u_0 = v_0 = 0.$$

Finally, estimates of $(\partial w/\partial r)_{r=1}$ at $\alpha = 0$ and $\alpha = \pi$ can be obtained by integrating (4) along the appropriate axis of symmetry from $r = 0$ to $r = 1$. After suitable reduction this gives

$$\left(\frac{\partial w}{\partial r}\right)_{r=1} = \int_0^1 \left(\frac{\partial \phi}{\partial \alpha}\frac{\partial w}{\partial r} - \frac{1}{r}\frac{\partial^2 w}{\partial \alpha^2}\right) dr - \tfrac{1}{2}D, \tag{49}$$

where the integral is evaluated along $\alpha = 0$ or $\alpha = \pi$ as the case may be. The integrand of the integral on the right-hand side of (49) is zero at $r = 1$ and hence the value of $(\partial w/\partial r)_{r=1}$ is not required in calculating the integral when numerical integration is used.

Values of $\tau(0)$ and $\tau(\pi)$ were obtained using the three methods for each of the three solutions for different h and k at each value of D and the convergence of the results with decreasing grid sizes was noted. In the case of $\tau(\pi)$ estimates were obtained in all three methods giving values of $D^{-1}\tau(\pi)$ which never differed from those given in table 4 by more than one unit in the third decimal place. For $\tau(0)$ there were small discrepancies in the three estimates for the higher values of D. It was found, however, that the estimates of $\tau(0)$ obtained using the second of the above methods based on central differences were converging upwards with decreasing grid sizes for a given D while those obtained from (49), using central differences to approximate all derivatives in the integral on the right, were converging downwards. The results given in table 4 represent an average of the corresponding two values obtained from the $h = 1/40$, $k = \pi/40$ solution. These are slightly lower at the higher values of D than the values given by Collins & Dennis (1975) who used only two terms of the backward-difference series of w at $r = 1$ for calculating $(\partial w/\partial r)_{r=1}$. More terms than this are necessary to give an accurate result.

One further consistency check on the boundary-layer structure can be made. Smith (1975) has considered, amongst other features of the boundary-layer flow, the stagnation point solution near $\alpha = 0$. From a knowledge of the axial velocity component at the edge of the boundary layer at $\alpha = 0$ it is possible from Smith's numerical integration of the boundary-layer equations to determine the axial component of skin friction at the wall. If we identify, on the boundary-layer scale, the axial velocity component at the edge of the boundary layer with the limit $D^{-\frac{3}{4}}w_M = 1.645$ as $D \to \infty$ given by (47) then, from the details of Smith's solution it is found that the corresponding limiting axial skin friction component at the wall is $D^{-1}\tau(0) = 0.819$ from the boundary-layer solution. This value gives excellent corroboration of the limiting trend of the corresponding axial skin friction in table 4.

6. Discussion

On the basis of results for one property only, the friction ratio γ_c/γ_s, Van Dyke (1978) has cast doubt on the previous boundary-layer analyses of flow in a slightly curved tube of circular cross-section and also on previous numerical results for this flow for $D > 1000$. The analysis of the present computations given in the previous section indicates that the results for γ_c/γ_s are not only completely consistent in themselves but that they are also consistent with the calculations of Collins & Dennis

(1975). Both sets of computations were carried out under the assumption that the coiling ratio $a/L = 0$, which is the assumption made by Van Dyke, so no explanation of any discrepancy in the results can be made on the basis of a difference in underlying assumptions. The calculations of Van Dyke (1978, p. 143) are not only in conflict for $D > 1000$ with those of Collins & Dennis (1975) but also with those of Truesdell & Adler (1970) and Austin & Seader (1973), both carried out for the case $a/L = 0.01$, and with the correlation of experimental results of Hasson (1955). The numerical results are in good agreement amongst themselves and also with the experimental correlation. In face of the body of agreement and the confirmation provided by the present results obtained by means of somewhat different methods of approximation to (4) and (5) from those used in previous numerical work together with the subsequent use of h^2 extrapolation, it must be concluded that the numerical work is not in error. In fact an error in γ_c/γ_s of the magnitude suggested by Van Dyke's predictions for this quantity is considered to be out of the question.

The present calculations are also consistent for other properties of the flow and they suggest strong evidence, as did the results of Collins & Dennis (1975), for the validity of some form of the conventional boundary-layer model as $D \to \infty$ even if some of the details remain to be settled. Van Dyke has criticized the previous numerical work on the grounds that the grid structures are too coarse at the higher values of D with only a few points in the boundary layer. If we assume that the boundary-layer thickness is indeed proportional to $D^{-\frac{1}{2}}$, as suggested by the results for $D^{\frac{1}{2}}(1-d)$ in table 3, we can provide some further consistency checks on the calculated results for γ_c/γ_s which tend to refute Van Dyke's criticism. For example, we would expect that the calculated result obtained using the grid $h = 1/20$, $k = \pi/20$ at $D = 500$ would be roughly of the same accuracy as the corresponding calculated result obtained using the grid $h = 1/40$, $k = \pi/40$ at $D = 4000$. We can test this hypothesis in relation to the calculated results for γ_c/γ_s given in table 1 in the following manner.

We denote as before by P_{123} a value in table 2 at a given D obtained by h^2 extrapolation from the corresponding values P_1, P_2 and P_3 of table 1. A relative error is then defined by

$$E_n = |P_n - P_{123}|/P_{123} \quad (n = 1, 2, 3). \tag{50}$$

If we apply this formula to the results for γ_c/γ_s at $D = 500$ we find $E_1 = 0.021$ and $E_2 = 0.006$. On the basis of a boundary-layer thickness proportional to $D^{-\frac{1}{2}}$ these values may be expected to be comparable, respectively, to the values of E_2 and E_3 at $D = 4000$. The actual values at $D = 4000$ are $E_2 = 0.026$, $E_3 = 0.007$. Similarly at $D = 1000$ we find the values $E_1 = 0.041$ and $E_2 = 0.012$ and it may be expected that the values of E_2 and E_3 at $D = 5000$ would be respectively somewhat smaller than these. In fact this is confirmed by the values $E_2 = 0.030$ and $E_3 = 0.008$ at $D = 5000$. The extrapolated value $\gamma_c/\gamma_s = 1.546$ at $D = 1000$ in table 2 is virtually in exact agreement with the corresponding result of Van Dyke (1978). In view of the above comparisons it is inconceivable that the extrapolated value $\gamma_c/\gamma_s = 2.383$ at $D = 5000$ in table 2 could be in error by more than 10 % as is suggested by the calculations of Van Dyke (1978).

It is difficult to speculate on reasons for the discrepancy between the present results and those of Van Dyke (1978). It seems unlikely that Van Dyke's calculations contain numerical errors in view of the agreement with the present results below $D = 1000$ and also on account of the excellent check between the first 14 terms of his series with

those calculated by Larrain & Bonilla (1970). There could of course be errors in the manipulation process to which the 24 calculated terms of the perturbation series are ultimately subjected, although the numerous tests of the process which are applied seem to be satisfactory. One possible very tentative reason for the discrepancy could be that Van Dyke's results represent a different solution to the problem from that given in the present paper. The recent work of Dennis & Ng (1980) indicates that non-uniqueness of solutions does appear to be possible in this problem as D increases and that it seems to start somewhere after $D = 1000$. In any case it would help to elucidate the discrepancy if Van Dyke's approach were extended to calculate properties of the flow other than simply the friction ratio. Such properties could then be compared with existing data, which might help to resolve the cause of the discrepancy.

REFERENCES

ADLER, M. 1934 *Z. angew. Math. Mech.* **14**, 257.

AKIYAMA, M. & CHENG, K. C. 1971 *Int. J. Heat Mass Transfer* **14**, 1659.

AUSTIN, L. R. & SEADER, J. D. 1973 *A.I.Ch.E. J.* **19**, 85.

BARUA, S. N. 1963 *Quart. J. Mech. Appl. Math.* **16**, 61.

COLLINS, W. M. & DENNIS, S. C. R. 1975 *Quart. J. Mech. Appl. Math.* **28**, 133.

COLLINS, W. M. & DENNIS, S. C. R. 1976*a* *J. Fluid Mech.* **76**, 417.

COLLINS, W. M. & DENNIS, S. C. R. 1976*b* *Proc. Roy. Soc.* A **352**, 189.

DEAN, W. R. 1927 *Phil. Mag.* **4**, 208.

DEAN, W. R. 1928 *Phil. Mag.* **5**, 673.

DENNIS, S. C. R. 1960 *Quart. J. Mech. Appl. Math.* **13**, 487.

DENNIS, S. C. R. & HUDSON, J. D. 1978 *Proc. 1st Int. Conf. on Numerical Methods in Laminar and Turbulent Flow*, p. 69. London: Pentech Press.

DENNIS, S. C. R. & NG, M. 1980 *22nd British Theoretical Mech. Coll., Univ. of Cambridge.*

GREENSPAN, D. 1973 *J. Fluid Mech.* **57**, 167.

HASSON, D. 1955 *Res. Correspondence* **1**, S1.

ITO, H. 1959 *Trans. A.S.M.E. D, J. Basic Engng* **81**, 123.

ITO, H. 1969 *Z. angew. Math. Mech.* **49**, 653.

LARRAIN, J. & BONILLA, C. F. 1970 *Trans. Soc. Reol.* **14**, 135.

McCONALOGUE, D. J. & SRIVASTAVA, R. S. 1968 *Proc. Roy. Soc.* A **307**, 37.

MORI, Y. & NAKAYAMA, W. 1965 *Int. J. Heat Mass Transfer* **8**, 67.

PATANKAR, S. V., PRATAP, V. S. & SPALDING, D. B. 1974 *J. Fluid Mech.* **62**, 539.

SMITH, F. T. 1975 *J. Fluid Mech.* **71**, 15.

SMITH, F. T. 1976 *Proc. Roy. Soc.* A **347**, 345.

TAYLOR, G. I. 1929 *Proc. Roy. Soc.* A **124**, 243.

TOPAKOGLU, H. C. 1967 *J. Math. Mech.* **16**, 1321.

TRUESDELL, L. C. & ADLER, R. J. 1970 *A.I.Ch.E.J.* **16**, 1010.

VAN DYKE, M. 1978 *J. Fluid Mech.* **86**, 129.

VARGA, R. S. 1962 *Matrix Iterative Analysis.* Prentice-Hall.

WHITE, C. M. 1929 *Proc. Roy. Soc.* A **123**, 645.

WOODS, L. C. 1954 *Aero. Quart.* **5**, 176.